VECTORS AND MATRICES IN TWO DIMENSIONS

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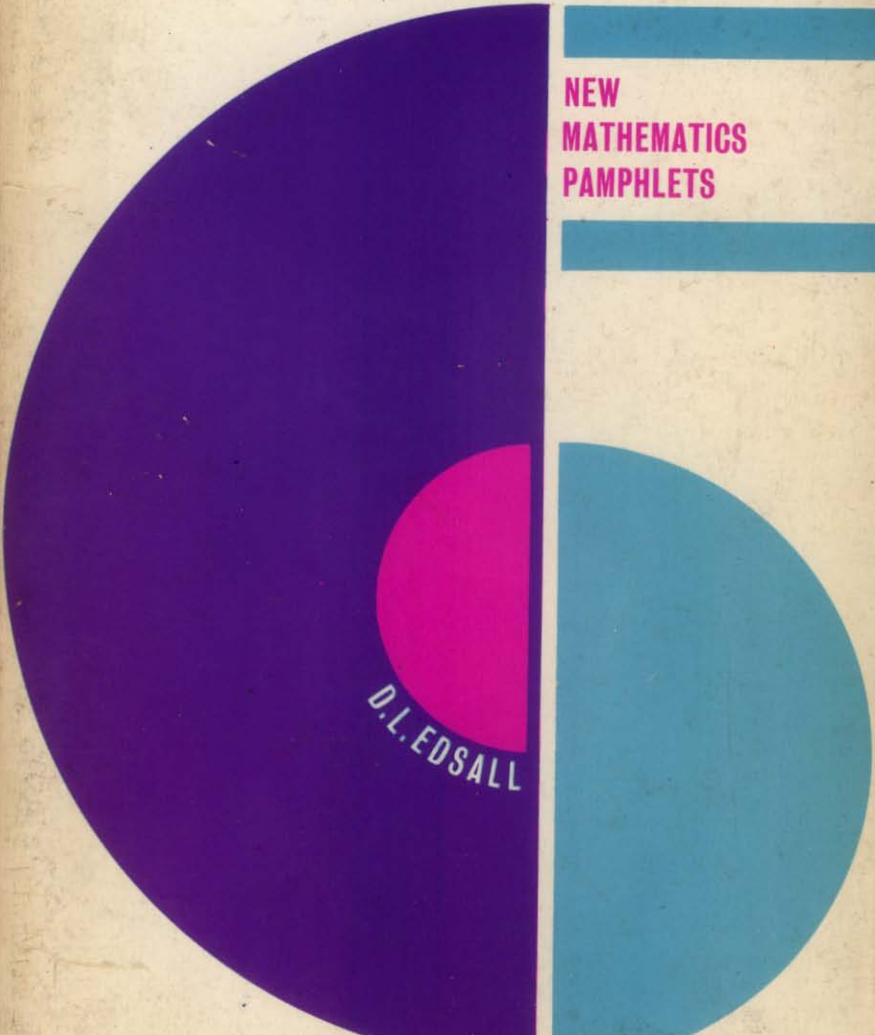

D. L. EDSALL

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GRAPHS AND INEQUALITIES

NEW MATHEMATICS PAMPHLETS

VECTORS AND MATRICES IN TWO DIMENSIONS

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Introduction

Matrices constitute surely one of the subjects new to the junior school syllabus that is most closely connected with the later development of many branches of mathematics and its applications. This pamphlet approaches matrix algebra from the viewpoint of a pupil who has already covered a certain amount of graphical work, and is also familiar with the normal methods of solving simultaneous equations. I have tried to show how the shorthand notation of matrix algebra is related to the familiar 'elementary' method that it represents. At the same time the important use of matrices for transformations in the plane of vectors has not been ignored.

Some of the questions have been marked with an asterisk. This indicates that, in an exercise where most of the questions are of a straightforward character, these questions require a greater understanding of the previous sections of the pamphlet. These questions may be used to extend the more able pupil, who will require more searching uses of the simple properties of the early sections of the pamphlet.

I would like to thank Mr. R. M. Dunnett for his assistance in checking the exercises, and reading the proofs of the text.

VECTORS AND MATRICES IN TWO DIMENSIONS

Mathematicians often discuss how objects or ideas are related to one another. If they want to talk about objects, they must have some simple way of describing their positions in space, and the opening sections of this pamphlet explain how this is done. An "object" may be quite complicated in outline, but most often its position (and the way it is related to the positions of other objects) can be determined if we can detail a few points which are part of the object. For example, to give your position in the room it would be sufficient to describe the point where one of the legs of your chair touches the ground. If all the class are sitting on the same kind of chair, then this one point would describe your relative positions in the room (at any rate if all the chairs face the front). We therefore start by trying to describe the position of a point in terms of mathematical concepts—in fact in terms of numbers.

The position of a point along a line

We will start by considering the easiest case, where all the points in which we are interested lie along a line.

We first take one point on the line; it will be called the *origin* and will always be denoted by the letter O. Its exact position on the line does not matter, but once it has been chosen it must not be changed (at any rate not without warning). We shall now describe the position of any other point P on the line by considering what kind of move we have to make in order to go from O to P. We shall denote such a move as OP, and OP will be called a *position vector* or just a *vector*. Notice that OP means just the length of the line between O and P, whereas OP represents the kind of move we have to make in going from O to P. In figure 1, OP = OQ, but clearly the move OP is not the same as the move OQ because they are in opposite directions. To help us to distinguish between the two, we shall call moves in the direction OP positive moves, and those in the direction OQ negative moves.

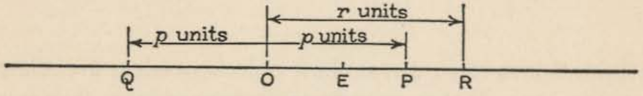

FIG. 1

But you should now be asking what justification there is for using a minus sign in this way. Well, let us consider what the number -5 means. The important equation is:

$$5+(-5)=0$$

that is that -5 is the number which "undoes" the act of adding the number 5; if you like it "brings you back to 0". Now let us consider our vectors. First we make a move OP, which takes us along the line from O to P; so to find the vector $-OP$ we want to find the move which will take us back to O from P (which could be written as PO). But OQ is a move of this kind —a move of length the same as OP but in the opposite direction. In order to use this fact we have to define a vector such that *two vectors will be considered equal if they are of the same length and are in the same direction.*

We can now put $PO = OQ$ and so write that $OQ = -OP$.

We still need a vector which is the total result of doing OP and then PO, i.e. the vector representing no move at all. This is usually written as O, because it behaves just like a nought, but has arrived as a result of thinking vectors. It is called the *zero vector*.

Clearly the two vectors OR and OP are different—for, although they are in the same direction, they are of different lengths. The length of a vector will be called its *magnitude*, and so OP is of magnitude p units and OR is of magnitude r units. The units chosen do not matter, but we must not change units without specifically saying so. When we choose a particular kind of unit to measure distances like OP, OQ, or OR, we are at the same time defining a particular vector OE which is one unit along the line from O in the positive direction. Such a vector is called a *unit vector* along the line. As this vector will appear frequently in the discussion of other vectors, we will use a special notation $\mathbf{i}$ for it.

Now the vector OP is a move in the positive direction which is p times the length of the unit vector (where OP is p units, see figure 1) and so we write that $OP = p\mathbf{i}$. Similarly $OR = r\mathbf{i}$ and $OQ = -p\mathbf{i}$ (minus because it is moving in the opposite direction to the unit vector $\mathbf{i}$). In general any vector OX can be written as $x\mathbf{i}$ where x is the distance OX measured in the same units as $\mathbf{i}$, together with a + or − sign depending on whether OX is in the same direction as $\mathbf{i}$ or in the opposite direction. Thus $\mathbf{i}$ tells us that the vectors are along the particular line we are considering, and the number x, together with its sign, tells us whereabouts we are along the line. If we now plot the vectors $(-4)\mathbf{i}$, $(-3)\mathbf{i}$, $(-2)\mathbf{i}$, $(-1)\mathbf{i}$, O, $\mathbf{i}$, $2\mathbf{i}$, $3\mathbf{i}$, $4\mathbf{i}$, we obtain the diagram shown in figure 2.

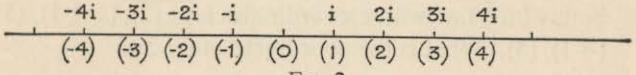

FIG. 2

Clearly if it is known that the points in which we are interested must all lie along this line, then the **i** part of the description is irrelevant, and so we just write the above vectors as (-4), (-3), (-2), (-1), (0), (1), (2), (3), and these numbers are called the *coordinates* of the points on the line. In general, the point with coordinate (x) is the point x units from 0, in a direction determined by the sign of x.

This method enables us to define the position of a point along a line, first by considering what kind of move is necessary to take us from a fixed point to the point in which we are interested, and then using a shorthand notation for this move, which is just a number (always written in brackets) called the coordinate of the point.

EXERCISE 1

1. Draw a diagram to illustrate the relative positions of the following vectors along a line: $5\mathbf{i}$, $(-0{\cdot}5)\mathbf{i}$, $2{\cdot}3\mathbf{i}$, $(-1{\cdot}7)\mathbf{i}$, $-\mathbf{i}$.

2. Draw a diagram to illustrate the relative positions of the points on a line whose coordinates are: (2), $(-1{\cdot}1)$, $(3{\cdot}2)$, (-1), (4). What is the coordinate of 0?

3. P is a point with coordinates (5) and Q is the point with coordinates (19). R is the mid-point of PQ. What is the coordinate of R? If P is (a) and Q is (b), and R is still the mid-point, find the coordinate of R in terms of a and b.

4. A is the point (a), B is the point (b), P is the point (p), and $\frac{\mathrm{AP}}{\mathrm{PB}} = \frac{3}{4}$. What is the value of p (i) if $a = 3$ and $b = 4$, (ii) if $a = -7$ and $b = 8$, (iii) if $a = -3$ and $b = -5$, and (iv) find a general formula for p in terms of a and b.

5. What is the resulting vector of doing the following moves one after the other? $3\mathbf{i}+5\mathbf{i}+(-4)\mathbf{i}+\mathbf{i}$.

6. Is there any difference between $-\mathbf{i}$ and $(-1)\mathbf{i}$?

7. Where are all the points on the line whose coordinates have values between (2) and (5)?

8. Where are all the points on the line whose coordinates are (i) greater than (4), and (ii) less than (-2)?

9. O is the origin, P is the point (i), X is a point between O and P such that $\frac{\mathrm{OX}}{\mathrm{XP}} = \frac{1}{4}$. What is the coordinate of X?

10. O is the origin and P the point (1). X is a point on PO produced such that $\frac{\mathrm{OX}}{\mathrm{PX}} = \frac{1}{4}$. What is the coordinate of X?

11. Repeat questions 9 and 10 where the value of $\frac{\mathrm{OX}}{\mathrm{XP}}$ (in 9) and $\frac{\mathrm{OX}}{\mathrm{PX}}$ (in 10) is p. Can p be greater than 1? What would be the implication of it being a minus number? What happens when $p = 1$?

***12.** O is the origin, P is the point (1), Q is the point (q), and R is the point (r). Find the value of $\frac{\mathrm{OP}\,.\,\mathrm{QR}}{\mathrm{OR}\,.\,\mathrm{QP}}$. (This is called the cross ratio of the four points OPQR, and is important in some branches of geometry.)

***13.** If the cross ratio of OPQR is equal to the cross ratio of ORQP, and $q = 3$, what are the possible values of r? (You should get two answers, one of which is impracticable.)

In the next three questions O, I, X, and Y are defined as follows:

O is the origin, I the point (2), X the point (x), and Y the point (y).

***14.** x and y are connected by the relation $xy = 4$. What happens to Y as X moves from I to O? What happens to Y when X gets beyond O and goes on moving in the same direction? If X is moving at a steady speed all the time, what can you say about the speed of Y?

***15.** x and y are connected by the relation $x+y = 4$. X moves as in question 14. What happens to Y? If X moves at a steady speed, what can you say about the speed of Y?

***16.** x and y are connected by the relation $xy+x+y = 8$. What happens when (i) X moves along the line, starting at I and moving steadily in the direction IO, (ii) X starts at I and moves steadily in the direction OI? What are the values of the coordinates of the points at which x and y coincide?

***17.** X and Y are points whose positions on a line are related. When X is at (O), Y is at (1). When X is at (1), Y is at (2). When X is at (2), Y is at (O). Find a general relation of kind that appears in question 16 which connects the coordinates x and y of any position of X and Y. (Consider a relation of the kind $axy+bx+cy+d = 0$ where you have to find a, b, c, and d.)

The position of a point in a plane

We now consider how the position of a point which is known to lie somewhere in a plane can be described. Practical examples would be the position of a player on a games field, or the position of a point on a piece of paper.

The basic method of describing the position of such a point is a development of the method we have already considered for a point along a line. We first select a point in the plane as an origin, which will still be denoted by O, and again we consider what kind of move we will have to make to go from O to any other point P in the plane. Again this is called a position vector (or just vector) and will be denoted by *OP*. The vector *OP* has a magnitude equal to the length of OP.

Notice that the term *direction of the vector OP* now has rather a different meaning. Consider figure 3; the points P, Q, R, and S all lie round a circle centre O, and so OP = OQ = OR = OS, but all the vectors *OP*, *OQ*, *OR*, and *OS* are quite different vectors for they clearly represent different kinds of ways of moving from O. As before, we shall say that $OP' = -OP$, because they are in opposite directions.

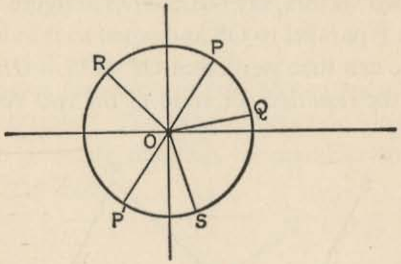

FIG. 3

Earlier we said that two vectors of the same magnitude and in the same direction would be called equal. We can now extend this to say *two vectors which are of the same magnitude and are parallel will be considered equal.*

If we have the origin O and two other points P and Q, then clearly a move from O to P is the same as a move from O to Q, followed by a second move from Q to P. Writing this in vectors we have that $OP = OQ + QP$. The effect is illustrated in figure 4.

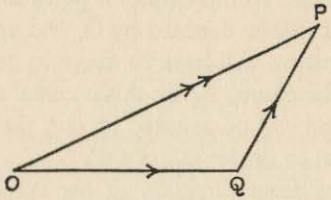

FIG. 4

This method is sometimes called the *triangle law for adding vectors*, as the sum of two vectors $OP + PQ$ can be found by drawing two sides of a triangle OQP and then constructing the third side to present the vector sum.

Since two parallel vectors of equal magnitudes are equal, we can add any two vectors, say OP and OS in figure 5, by drawing a vector from P parallel to OS and equal to it in length and so form PX. We can then write that $OP + OS = OP + PX = OX$. OX is called the *resultant* (or sum) of the two vectors OP and OS.

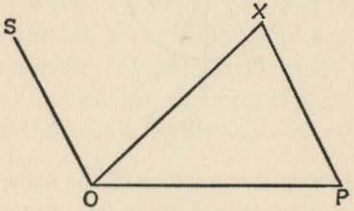

FIG. 5

It is important to realize that although we have written $OP + PQ$ for the resultant of OP and PQ, the + sign is being used for a slightly different process to that of addition in arithmetic. Sometimes the resultant is called the *sum* (or the *vector sum*) of the vectors, but again we must realize that this is using the word "sum" in a different sense from the "sum" of two numbers in arithmetic.

Although we have talked up till now of "position vectors" or "moves from P to Q", many other physical and mathematical phenomena have this same vector characteristic—that is both a magnitude and a direction* (or their equivalents) must be known before the "thing" is completely defined. When this is the case the triangle law of addition must always be used. Other examples of vectors are velocities (e.g. of an aeroplane, or an athlete on a track), accelerations (e.g. of a car, or that due to gravity), forces (e.g. the tension in a tow-rope); consequently physicists and engineers, as well as mathematicians, are continually talking about vectors. Other applications of vector ideas will appear later on.

EXERCISE 2

1. Figure 6 shows two vectors OP, OQ of lengths 2 and 3 units respectively. The angle between them is 47°. Produce an accurate diagram to calculate the sum (or resultant) $OP + OQ$.

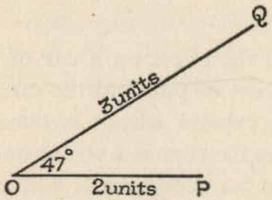

FIG. 6

2. In figure 7 OX and OY are two vectors at right angles. What is the magnitude of their resultant?

* It is not generally necessary to define the *position* of a vector.

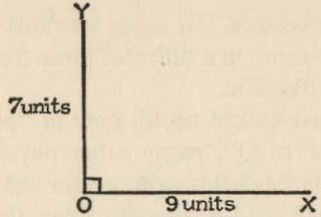

FIG. 7

3. Using the vectors of question 1, draw an accurate diagram to find the vector $OP-OQ$.

4. ABC is a triangle. P is the midpoint of AB, and Q is the midpoint of AC. By considering the vector sum $BA+AC$, and also the vector sum $PA+AQ$, show that PQ is parallel to BC and also BC = 2PQ. Remember that two vectors that are equal must be both equal in magnitude and parallel.

5. Find the resultant of the pair of vectors OP and OQ, where $OP=4$ units and OQ is 5 units and the angle POQ = 127°. (You may do this by drawing an accurate diagram.)

6. Find the vector $OP-OQ$ for the two vectors OP and OQ in question 5.

7. By considering the effect on a car of a tow-rope, where both the direction of pull and the magnitude of the pull can be varied, explain why it is reasonable to suppose that the force in the rope is a vector quantity. If two men are pulling the car along, each with his own tow-rope, explain how you could work out in which direction the car would move as a result of their combined efforts. (You may assume that there is no one operating the steering wheel!)

8. A peg is placed at the origin, and attached to it are two strings pulled by two men. The first pulls with a force of 3 lbf and the second with a force of 7 lbf. The angle between the strings is 67°. A third man also has a string attached to the peg. In what direction and with what force should he pull in order to balance the other two men?

9. An aeroplane is facing north and its engines push it forward at 400 mile/h. A wind blows the aeroplane due east at 50 mile/h. In what direction and at what speed does it actually fly?

10. A river is 2 miles wide. A boat can travel at 5 mile/h, and the current is flowing at 3 mile/h. The man in the boat wants to travel straight across the river. In which direction should he point the boat, and how long will it take him to cross?

***11.** Prove the following geometrical properties by using triangles of vectors:

(i) The diagonals of a parallelogram bisect each other.

(ii) If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.

(iii) If ABCD is any quadrilateral, and P, Q, R, and S are the midpoints of AB, BC, CD, and DA respectively, prove that PQRS is a parallelogram.

(For this question you should form vector sums using triangles of vectors. Remember that if two vectors are equal, they must be both equal in magnitude and parallel.)

We will now use the idea of splitting a vector up into two separate vectors to find a description of the position of a point in a plane similar to the coordinate of a point on a line. First we notice that if we take any vector OP we can draw any two lines, one through O and one through P (assuming that they are not parallel) and let them cut at M. Then straightaway we have that $OP = OM + MP$. The two vectors OM and MP are called two *components* of OP.

We will now consider a special case where the two components into which we split OP are at right angles. We do this to simplify the calculations. We could use lines which are not at right angles, and most of the results would still be true, but the calculations would be very complicated. We therefore draw two lines at right angles through O. It is the convention to draw one horizontally X′OX in figure 8, and the other Y′OY vertically. These lines will be called the *axes*, and in particular X′OX will be called the *x-axis* and Y′OY the *y-axis*.

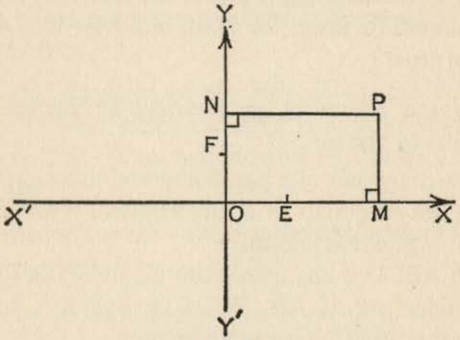

FIG. 8

We now split the vector OP into its two components OM and MP. Notice that since ON and MP are both equal and parallel (being opposite sides of the rectangle OMPN) we can write that $MP = ON$ and so $OP = OM + ON$. Moreover there is only one line through P parallel to Y′OY, and only one line through P parallel to X′OX, and so there is only one point M and one point N corresponding to any one point P. We can now consider how to describe the position of M on X′OX and N on Y′OY. But this is what we did in the first section of the book; now it is necessary to do it twice over. We first select a unit vector OE on X′OX and call it $\mathbf{i}$, then if OM measures m units we can write $OM = m\mathbf{i}$. Similarly we select OF a unit vector along Y′OY, and call it $\mathbf{j}$. If ON is n units, then we write that $ON = n\mathbf{j}$. Then

$$OP = OM + MP = OM + ON = m\mathbf{i} + n\mathbf{j}$$

We cannot simplify this last expression any more, as there is no connection between $\mathbf{i}$ and $\mathbf{j}$; for if $\mathbf{j} = g\mathbf{i}$ then that would mean that $\mathbf{j}$ was a vector along X′OX, which we know is not true. Again this means that instead of one number which we can use as a coordinate for the point, we now have two, one associated with the x-axis and the other with the y-axis. They are written as (m, n) with the one associated with the x-axis (which is called the *x-coordinate*) always written before the one associated with the y-axis (called the *y-coordinate*).* This

---

* There are times when this twofold meaning of the number pair is confusing. In order to overcome any ambiguity we shall always use (m, n) to mean the coordinates of a point, i.e. it is just the description of the position of a point in a plane. If, on the other hand, we wish to talk specifically about the vector OP, i.e. the actual move from O to P, then it will be written as $\begin{pmatrix} m \\ n \end{pmatrix}$. For example, if we want a vector in the same direction as the vector $\begin{pmatrix} m \\ n \end{pmatrix}$ but three times as long we have

$$\begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 3m \\ 3n \end{pmatrix}$$

and this may be written as $3\begin{pmatrix} m \\ n \end{pmatrix}$. On the other hand, the expression $3(m, n)$ has no meaning. Notice that in all cases it is essential to think of the numbers in the correct order, and that if it is a vector written as $\begin{pmatrix} m \\ n \end{pmatrix}$ then the top number is the first of the pair.

pair of coordinates form what is called an "ordered pair of numbers"; for it is easy to see from figure 9 how important is the order in which the coordinates are written down, and that $(1,3)$ is not at all the same point as $(3,1)$.

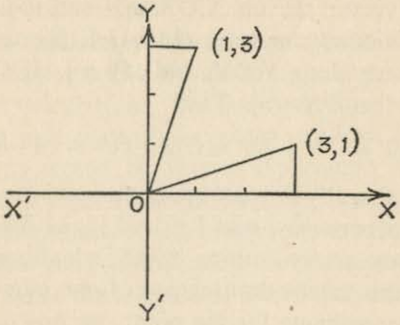

FIG. 9

Notice that the position of the point is primarily given by the vector OP, and that this in turn can be represented by the coordinates (m,n). Consequently (m,n) can also be used to describe the vector OP as well as just detailing the position of a point in a plane.

We will now consider the triangle rule for adding vectors for components and ordered pairs as methods of describing vectors. We will suppose that we want to form the vector sum $OP+OS$ (see figure 10). First we draw PQ as a vector parallel to OS and equal to it in magnitude. Since PQ is equal to OS we now have to consider $OP+PQ$, the answer to which should be OQ. Now suppose that OP is $\begin{pmatrix}4\\2\end{pmatrix}$ and $OS\begin{pmatrix}2\\3\end{pmatrix}$, so that if **i** and **j** are unit vectors as before we have that $OP=4\mathbf{i}+2\mathbf{j}$ and $OS=PQ=2\mathbf{i}+3\mathbf{j}$.

So

$$OP+OS=(4\mathbf{i}+2\mathbf{j})+(2\mathbf{i}+3\mathbf{j})=(4+2)\mathbf{i}+(2+3)\mathbf{j}=6\mathbf{i}+5\mathbf{j}$$

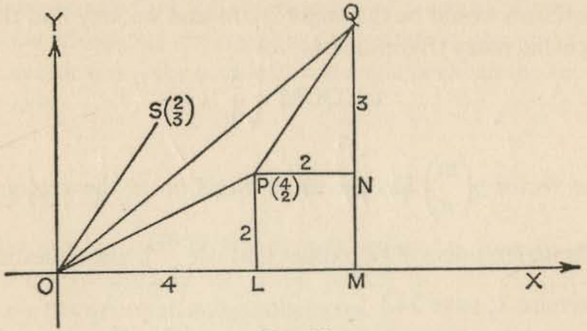

FIG. 10

The addition is therefore done on each component on its own. The diagram shows that it represents two simultaneous additions:

$$OL+PN=OM \quad \text{and} \quad LP+NQ=MQ$$

If we write this in ordered-pair notation we get that

$$\begin{pmatrix}4\\2\end{pmatrix}+\begin{pmatrix}2\\3\end{pmatrix}=\begin{pmatrix}6\\5\end{pmatrix}$$

Clearly OQ is the vector $\begin{pmatrix}6\\5\end{pmatrix}$

This idea of splitting vectors up into components first and then doing any additions which are wanted, and finally putting the two components back into a single vector, is a basic method of handling most practical examples of vector additions.

It is sometimes convenient to describe a vector as a magnitude together with the direction which it makes with the x-axis (in the direction in which x increases). In the case above the magnitude is the length of OQ, which may be found by using Pythagoras' Theorem on the triangle OQM:

$$OQ=\sqrt{(6^2+5^2)}=\sqrt{61}$$

Its direction would be the angle QOM and we may find this by using elementary trigonometry, for

$$\tan \mathrm{QOM} = \frac{5}{6}$$

The vector $p\begin{pmatrix}m\\n\end{pmatrix}$ has the same direction as the vector $\begin{pmatrix}m\\n\end{pmatrix}$ and its magnitude will be p times that of $\begin{pmatrix}m\\n\end{pmatrix}$. (See question 14 of Exercise 3, page 24.)

Elementary trigonometry may be used in other ways, for if *OP* is a vector and its two components are *OM* and *ON* (which is equal to *MP*), then (figure 11)

$$\frac{\mathrm{OM}}{\mathrm{OP}} = \cos\theta, \quad \text{i.e.} \quad \mathrm{OM} = \mathrm{OP}\cos\theta$$

therefore $$\frac{\mathrm{MP}}{\mathrm{OP}} = \sin\theta, \quad \text{i.e.} \quad \mathrm{MP} = \mathrm{OP}\sin\theta$$

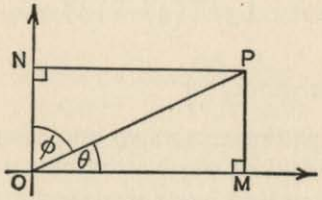

FIG. 11

The two components of OP may therefore be written as

$$OP = (\mathrm{OP}\cos\theta)\mathbf{i} + (\mathrm{OP}\sin\theta)\mathbf{j}$$

or alternatively

$$\frac{\mathrm{ON}}{\mathrm{OP}} = \cos\phi \quad \text{and so} \quad \mathrm{ON} = \mathrm{OP}\cos\phi$$

We can say that the components of a vector in two perpendicular directions are equal in magnitude to the magnitude of the vector times the cosine of the angle between the vector and the axis.

EXERCISE 3

Solutions to these problems may either be done by accurate drawing (preferably on graph paper) or else by calculation, unless the question states otherwise.

1. Find the magnitude of the vector which is the sum of $(3\mathbf{i}+7\mathbf{j})$ and $(6\mathbf{i}+7\mathbf{j})$.

2. Find the magnitude and direction (given the angle between the vector and the x-axis) of the vector which is the vector sum of $(3, 5)$ and $(2, 8)$.

3. Repeat question 1 with the vectors $(2\mathbf{i}-5\mathbf{j})$ and $(-3\mathbf{i}-5\mathbf{j})$.

4. Repeat question 2 with the vectors $\begin{pmatrix}-1\\4\end{pmatrix}$ and $\begin{pmatrix}3\\-2\end{pmatrix}$.

5. Find the magnitude and direction of the vector $\begin{pmatrix}4\\3\end{pmatrix}-\begin{pmatrix}2\\-5\end{pmatrix}$.

6. *OP* is a vector of magnitude 8 units in a direction which makes an angle of 35° with the x-axis. Find its two components along the x-axis and y-axis.

7. Find in terms of **i** and **j** the vector

$$(4\mathbf{i}-5\mathbf{j})+(-3\mathbf{i}+8\mathbf{j})-(2\mathbf{i}+7\mathbf{j})$$

and illustrate your answer with a diagram.

8. Three vectors are as follows: OP of magnitude 6 at 20° to the x-axis, OQ of magnitude 4 at 32° to the x-axis, and OR of magnitude 5 at 57° to the x-axis. Find the magnitude and direction of the sum $OP+OQ+OR$.

9. Three vectors are $\begin{pmatrix}4\\-7\end{pmatrix}$, $\begin{pmatrix}-1\\5\end{pmatrix}$, and $\begin{pmatrix}4\\-2\end{pmatrix}$. Find the magnitude and direction of their resultant.

10. Find the magnitude and direction of the vector

$$3\begin{pmatrix}1\\2\end{pmatrix}+2\begin{pmatrix}1\\3\end{pmatrix}.$$

11. Find values of x and y such that $\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}3\\7\end{pmatrix}=2\begin{pmatrix}5\\1\end{pmatrix}$.

12. Find values of x and y such that $2\begin{pmatrix}x+3\\y+5\end{pmatrix}=\begin{pmatrix}7\\2\end{pmatrix}$.

13. Find the magnitude and direction of the vector $\begin{pmatrix}x\\y\end{pmatrix}$ if

$$3\begin{pmatrix}x+1\\y-2\end{pmatrix}=\begin{pmatrix}15\\3\end{pmatrix}.$$

14. Prove the result stated on p. 22 that the magnitude of $p\begin{pmatrix}m\\n\end{pmatrix}$ is p times that of $\begin{pmatrix}m\\n\end{pmatrix}$, and that its direction is the same as that of $\begin{pmatrix}m\\n\end{pmatrix}$.

15. Find the angle between the two vectors (i) $\begin{pmatrix}5\\4\end{pmatrix}$ and $\begin{pmatrix}7\\9\end{pmatrix}$, (ii) $\begin{pmatrix}5\\4\end{pmatrix}$ and $\begin{pmatrix}-7\\9\end{pmatrix}$, (iii) $\begin{pmatrix}5\\4\end{pmatrix}$ and $\begin{pmatrix}7\\-9\end{pmatrix}$, (iv) $\begin{pmatrix}5\\4\end{pmatrix}$ and $\begin{pmatrix}-7\\-9\end{pmatrix}$.

We will now see how these vector ideas may be used to investigate some of the geometrical properties of points and lines. The basic method of approach is always to split any move in which we are interested into its two components parallel to the axes. (For this action we shall assume that the axes are always taken at right angles to each other.) In figure 12, PQ is a given line. By a given line we mean that its position is completely determined, and the way that this has been done in this case is by detailing the positions of two points that lie on the line. P is (2, 5) and Q is (3, 7). By the line PQ we shall mean not only the section which lies between P and Q, but the whole line which is obtained by producing PQ indefinitely in both directions.

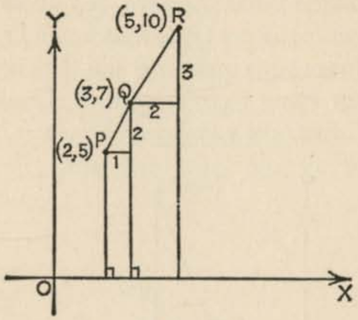

FIG. 12

The first question that arises is, if we are given another point, say R with coordinates (5, 10), does this point also lie on the line? We tackle this question by considering what kind of vectors are PQ and QR, and specially whether they are in the same direction. If they are, then QR is clearly an extension of PQ, and so P, Q, and R all lie along the line. What kind of move is PQ? From the diagram it is not difficult to see that we have to move $(3-2)$ units in the x-direction (i.e. PQ has a component of 1 unit in the x-direction) and $(7-5)$ units in the y-direction (i.e. PQ has a component of 2 units in the y-direction). Now, similarly, QR has an x-component of $(5-3) = 2$ units, so if we are to stay along the same direction as PQ it should have a y-component twice that of PQ, i.e. it should have a y-component of $2 \times 2 = 4$ units. But the y-component actually is $(10-7) = 3$ units, and so R does not lie on the line PQ.

We are likely to come across many straight lines, and it is useful to find some kind of condition that will tell us quickly whether a point does or does not lie on a particular straight line. As we are describing all our points in terms of numbers, the condition that we shall get will be an algebraic one. The work is precisely the same as above, but we shall work with a point whose coordinates are (x, y) where x and y are any suitable numbers. Again we will consider P and Q of figure 12, and PQ has x-component 1 and y-component 2. The x-component of QS is now $(x-3)$ and the y-component is $(y-7)$ (see figure 13).

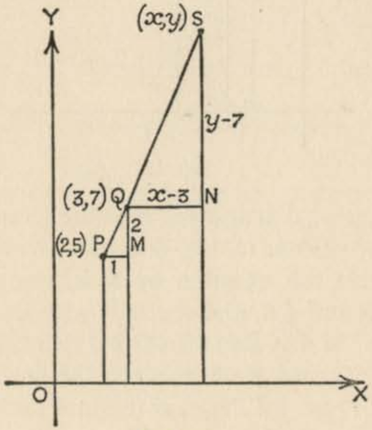

FIG. 13

For PQS to be a straight line, triangles PQM, QSN must be similar, and so we have that the y-component $(y-7)$ must be $\frac{2}{1}$ of the x-component $(x-3)$,

i.e.
$$y-7 = \frac{2}{1}(x-3)$$
$$y-7 = 2x-6$$
$$y = 2x+1$$

This last equation is called *the equation of the line* PQS.

The method of resolving movements along a line into two components in the x-direction and y-direction, and then comparing the triangles formed, will help to solve many problems connected with straight lines. This basic method should be used to solve the following problems. Not all of them use precisely the same arguments that we have described, but they can be solved by using the first basic step of resolving into components, together with one or more elementary geometrical ideas.

EXERCISE 4

1. Does the point $(3, 7)$ lie on the line through $(2, 5)$ and $(5, 11)$?

2. Does the point $(4, 5)$ lie on the line through $(1, 17)$ and $(3, 9)$?

3. Do the following sets of three points all lie on a line?

(i) $(1, 7)$, $(-3, 15)$, $(6, -1)$

(ii) $(-3, 7)$, $(2, 3)$, $(8, 14)$

4. Find the values of h in order that the three points may lie in a straight line in each case

(i) $(1,7), (-1,5), (3,h)$
(ii) $(3,5), (-4,2), (h,6)$
(iii) $(-2,11), (4,h), (-7,8)$.

5. Find the distance between the points $(2,7)$ and $(5,11)$.

6. Find the distance between the points $(-3,6)$ and $(2,9)$.

7. Find the length of the vector $\begin{pmatrix}2\\5\end{pmatrix}+\begin{pmatrix}6\\9\end{pmatrix}$.

8. Find the coordinates of the mid-points of the following pairs of points (i) $(1,5)$ and $(3,4)$; (ii) (a,b) and (c,d).

9. The reflection of a point P in a line is defined as follows: A line is drawn through P, perpendicular to the given line and meeting it at Q. PQ is produced to R so that PQ = QR. R is then the reflection of P in the line. If P is the point $(3,4)$ what are the coordinates of its reflection in (i) the x-axis, (ii) the y-axis, (iii) the line through O at 45° to the x-axis?

10. A is the point $(1,7)$. B is the point $(6,9)$. C is the point $(3,5)$. D is a point such that ABCD is a parallelogram. What are the coordinates of D?

11. P is the point $(2,7)$. Q is the point $(3,10)$. R is the point $(5,16)$. What is the value of $\frac{PQ}{QR}$?

12. P is the point $(3,4)$. Q is the point $(2,-1)$. R is the point $(5,14.)$ What is the value of $\frac{PR}{QR}$?

13. P is the point $(2,1)$. Q is the point $(3,3)$. R is the point $(4,2)$. X is the mid-point of QR. Y is a point on PX such that $\frac{PY}{PX}=\frac{2}{3}$, what are the coordinates of Y?

14. P is $(2,7)$, Q is $(4,9)$, R is $(5,12)$, and S is $(8,a)$. PQ is parallel to RS. Find the value of a.

In the last section we saw that those points represented by pairs of values (x,y), such that the point (x,y) always lay on a certain straight line, were given by $y=2x+1$. Any expression of this kind (so many x's, so many y's, and a constant) can be shown to represent the condition for a point to lie on some straight line. For example

$$\begin{aligned} 4y &= 9x-1 \\ &= 9(x-1)+8 \\ \text{therefore}\qquad 4y-8 &= 9(x-1) \\ 4(y-2) &= 9(x-1) \\ \text{therefore}\qquad \frac{y-2}{x-1} &= \frac{9}{4} \end{aligned}$$

By comparison with the work on p. 27, this is the condition for a line which passes through the point (1, 2) and moves 4 in the x-direction as compared with 9 in the y-direction.

If we draw the graphs of the equations $y=2x+1$ and $4y=9x-1$ we get a diagram like figure 14.

We now want to find the coordinates of the point P where the lines cut. In order to do this we notice that all the points on PQ satisfy $4y=9x-1$, and all the points on PR satisfy $y=2x+1$.

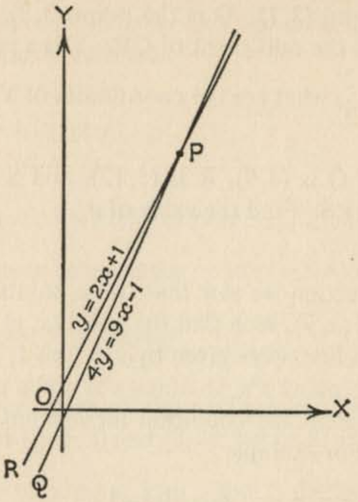

FIG. 14

Since P lies on both—and it is the only point which does—its coordinates must satisfy both equations simultaneously. This indicates that we must solve the simultaneous equations

$$\begin{aligned} y &= 2x+1 \\ 4y &= 9x-1 \end{aligned}$$

There are several methods of doing this; the one in which we are interested is aimed at finding the vector *OP* where O is the origin. We shall consider what algebraic changes we may make to the original equations to obtain the solution. The two main steps are:

(i) We may multiply either equation by any number (other than 0).

(ii) We may add or subtract the two equations.

The solution will be obtained by considering just the coefficients on their own. We start by writing the equations as

$$\begin{aligned} 2x-y &= -1 \\ 9x-4y &= 1 \end{aligned} \quad \text{and then shorten this to} \quad \begin{array}{cc|c} 2 & -1 & -1 \\ 9 & -4 & 1 \end{array}$$

Notice that it is most important that the numbers should be written down in the correct order—in each row the coefficient of x appears first, that of y next, and then separately the constant on "the other side" of the equals sign. If you consider the normal method of solution of a pair of simultaneous equations, then this is done automatically by always keeping the x's and y's under each other.

We now consider the rules (i) and (ii) above and see how they may be applied to our array of numbers:

(i) We may multiply either row of numbers by any number (except 0). For example starting with $\begin{array}{cc|c} 2 & -1 & -1 \\ 9 & -4 & 1 \end{array}$ we may multiply the first row by 3 to get $\begin{array}{cc|c} 6 & -3 & -3 \\ 9 & -4 & 1 \end{array}$ and this new array will still represent a pair of equations with the same solution.

(ii) We may add or subtract our rows of numbers to form a new row. Thus if we start again with the original array $\begin{array}{cc|c} 2 & -1 & -1 \\ 9 & -4 & 1 \end{array}$ and we form a new first row by adding the two rows together we get $\begin{array}{cc|c} 11 & -5 & 0 \\ 9 & -4 & 1 \end{array}$ and this again will represent a pair of equations which will have the same solution as the pair with which we started.

Our aim is to use combinations of these two rules in order to change our array of numbers into the form $\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b \end{array}$ for then we would have the equations $\begin{matrix}x=a\\y=b\end{matrix}$ or in other words the solution. This is done steadily one step at a time, as shown below, each step giving one of the numbers wanted in the final array.

$$\begin{array}{lrr|rlrr}
\hline
 & 2 & -1 & -1 & & & \\
 & 9 & -4 & 1 & & & \\
\hline
\text{(1st row)}\div 2 & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} & \text{in order to get} & 1^{*} & 0 \\
 & 9 & -4 & 1 & & 0 & 1 \\
\hline
 & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} & \text{in order to get} & 1 & 0 \\
\text{(2nd row)}-9\times\text{(1st row)} & 0 & \tfrac{1}{2} & 5\tfrac{1}{2} & & 0^{*} & 1 \\
\hline
 & 1 & -\tfrac{1}{2} & -\tfrac{1}{2} & \text{in order to get} & 1 & 0 \\
\text{(2nd row)}\times 2 & 0 & 1 & 11 & & 0 & 1^{*} \\
\hline
\tfrac{1}{2}\text{(2nd row)}+\text{(1st row)} & 1 & 0 & 5 & \text{in order to get} & 1 & 0^{*} \\
 & 0 & 1 & 11 & & 0 & 1 \\
\hline
\end{array}$$

and so the solution is $x=5$, $y=11$, or in other words OP is the vector $\begin{pmatrix}5\\11\end{pmatrix}$. Notice where the vector answer appears in the final column of numbers.

This form of solution is much better expressed in vectors right from the beginning. This is done as follows:

The original equations $\begin{matrix}2x-y=-1\\9x-4y=1\end{matrix}$ are written as $\begin{pmatrix}2&-1\\9&-4\end{pmatrix}\cdot\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}$ where $\begin{pmatrix}x\\y\end{pmatrix}$ is the vector that we want to find, $\begin{pmatrix}-1\\1\end{pmatrix}$ is a vector formed from the numbers in the original equations. $\begin{pmatrix}2&-1\\9&-4\end{pmatrix}$ is the array of numbers which are the coefficients in their correct places. Such an array is

called a *matrix*, and it can be thought of as two vectors put side by side and working as one unit. Each number in the matrix is called an *element*. The process is a form of multiplication defined between matrices and vectors (and later between matrices and matrices) which is in effect a rule for obtaining the left-hand side of the original equations from the matrix. The process is as follows:

(i) Take each element of the first row of the matrix in turn and multiply it by the corresponding element of the vector. Then add these products together to give the first element of the resulting vector.

(ii) Repeat this process with the elements of the second row of the matrix.

This process is then

$$\begin{pmatrix}2&-1\\9&-4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}\Rightarrow\left(\begin{matrix}2\times x+\\9\times x+\end{matrix}\begin{pmatrix}-1\\-4\end{pmatrix}\begin{matrix}\times y\\\times y\end{matrix}\right)\Rightarrow\begin{pmatrix}2x-y\\9x-4y\end{pmatrix}$$

where the sign ⇒ means "implies that". This method of "multiplication" is called *matrix multiplication* and will be written in the same way as that for ordinary numbers. It is important to realize that this is different from ordinary arithmetical multiplication.

We now look again at our solution of the simultaneous equations to see the part that this matrix multiplication plays. Two new columns of arrays are used. Column I starts with $\begin{pmatrix}1&0\\0&1\end{pmatrix}$ and at each step is changed in the same way as the left-hand column. Column E shows the effect of each change on the array $\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

				I		E	
	2	−1	−1	1	0	1	0
	9	−4	1	0	1	0	1
(1st row)÷2	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	0
	9	−4	1	0	1	0	1
	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0
(2nd row) − (1st row × 9)	0	$\frac{1}{2}$	$5\frac{1}{2}$	$-\frac{9}{2}$	1	−9	1
	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	1	0
(2nd row) × 2	0	1	11	−9	2	0	2
(1st row) + (2nd row × $\frac{1}{2}$)	1	0	5	−4	1	1	$\frac{1}{2}$
	0	1	11	−9	2	0	1

The first matrix to look at is $\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}$ which is the final result in column *I*. Now

$$\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}=\left[\begin{pmatrix}-4\\ -9\end{pmatrix}\begin{matrix}\cdot\\ \cdot\end{matrix}\begin{pmatrix}-1\\ -1\end{pmatrix}\begin{matrix}+1.1\\ +2.1\end{matrix}\right]=\begin{pmatrix}5\\ 11\end{pmatrix}$$

or in other words if the vector of numbers $\begin{pmatrix}-1\\ 1\end{pmatrix}$ from the original equations is multiplied by the end matrix in column *I* we get the answer vector $\begin{pmatrix}5\\ 11\end{pmatrix}$. Is this just pure chance, or can we find a reason?

The original equation was $\begin{pmatrix}2 & -1\\ 9 & -4\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$, and suppose that we can use matrix multiplication to multiply each side by the matrix $\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}$. Then we have that

$$\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\cdot\begin{pmatrix}2 & -1\\ 9 & -4\end{pmatrix}\cdot\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\cdot\begin{pmatrix}-1\\ 1\end{pmatrix}=\begin{pmatrix}5\\ 11\end{pmatrix}$$

i.e.
$$\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\left[\begin{pmatrix}2 & -1\\ 9 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}\right]=\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\begin{pmatrix}2x-y\\ 9x-4y\end{pmatrix}$$
$$=\begin{pmatrix}(-4)(2x-y)+(1)(9x-4y)\\ (-9)(2x-y)+(2)(9x-4y)\end{pmatrix}$$
$$=\begin{pmatrix}-8x+4y+9x-4y\\ -18x+9y+18x-8y\end{pmatrix}$$
$$=\begin{pmatrix}x\\ y\end{pmatrix}$$

which explains why multiplication by this matrix gave the solution to the equations. If we examine the last matrix of the third-last line of the equation we see that

$$\begin{bmatrix}(-4)(2x-y)+(1)(9x-4y)\\ (-9)(2x-y)+(2)(9x-4y)\end{bmatrix}$$

may be written as

$$\begin{bmatrix}(-4.2+1.9)x+(-4.-1+1.-4)y\\ (-9.2+2.9)x+(-9.-1+2.-4)y\end{bmatrix}$$

Again this may be written as

$$\begin{bmatrix}(-4.2+1.9)(-4.-1+1.-4)\\ (-9.2+2.9)(-9.-1+2.-4)\end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

and closer examination of this last matrix shows it to be made up by using matrix multiplication with $\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\begin{pmatrix}2\\ 9\end{pmatrix}$ and $\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\begin{pmatrix}-1\\ -4\end{pmatrix}$ and then writing the resulting two vectors side by side. Thus we may use matrix multiplication to "multiply" two matrices together if we use the rules for multiplying a matrix and a vector, and think of the second matrix as two vectors alongside each other. Thus

$$\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\left(\begin{array}{c|c}2 & -1\\ 9 & -4\end{array}\right) = \left[\begin{array}{c|c}(-4)2+1.9 & (-4)(-1)+1.(-4)\\ (-9)2+2.9 & (-9)(-1)+2.(-4)\end{array}\right]$$

$$= \left(\begin{array}{c|c}1 & 0\\ 0 & 1\end{array}\right) = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

Consequently the left-hand side of our equation simplifies to

$$\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\ 9 & -4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}x\\ y\end{pmatrix}$$

and so we can get the solution. In other words the matrix $\begin{pmatrix}-4 & 1\\ -9 & 2\end{pmatrix}$ "unscrambles" the matrix of the equation. Before we investigate this result any further, you should get used to handling matrix multiplication.

EXERCISE 5

1. Write down the following equations as a normal pair of simultaneous equations (you are not asked to solve them):

(i) $\begin{pmatrix}3 & -1\\ -2 & 4\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}7\\ 2\end{pmatrix}$ (ii) $\begin{pmatrix}7 & 2\\ 5 & -2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}11\\ 5\end{pmatrix}$

2. Find the vectors which are obtained as a result of the following matrix multiplications:

(i) $\begin{pmatrix}3 & 4\\ -2 & 5\end{pmatrix}\cdot\begin{pmatrix}5\\ 3\end{pmatrix}$ (ii) $\begin{pmatrix}5 & 7\\ 2 & -1\end{pmatrix}\cdot\begin{pmatrix}3\\ -2\end{pmatrix}$

(iii) $\begin{pmatrix}2 & 4\\ 6 & 7\end{pmatrix}\cdot\begin{pmatrix}-3\\ 1\end{pmatrix}$ (iv) $\begin{pmatrix}6 & 4\\ 3 & 2\end{pmatrix}\cdot\begin{pmatrix}2\\ -3\end{pmatrix}$

3. Find the matrix which is the product of the following pairs of matrices:

(i) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\cdot\begin{pmatrix}1 & 3\\ 5 & 7\end{pmatrix}$ (ii) $\begin{pmatrix}-2 & 3\\ 1 & -2\end{pmatrix}\cdot\begin{pmatrix}5 & 2\\ 2 & 1\end{pmatrix}$

(iii) $\begin{pmatrix}1 & 7\\ 1 & 7\end{pmatrix}\cdot\begin{pmatrix}2 & 5\\ 2 & 5\end{pmatrix}$ (iv) $\begin{pmatrix}3 & 7\\ 7 & 3\end{pmatrix}\cdot\begin{pmatrix}3 & 7\\ 7 & 3\end{pmatrix}$

4. Two boys go into a shop to buy sweets. What they buy is written down like this:

	Wine gums	*Aniseed balls*
John	3 oz	2 oz
Mike	2 oz	4 oz

The prices are:
Wine gums 2d per oz
Aniseed balls 1d per oz

If we form the matrix product $\begin{pmatrix}3 & 2\\ 2 & 4\end{pmatrix}\cdot\begin{pmatrix}2\\ 1\end{pmatrix}$, what does the resulting vector represent?

5. One kind of concrete requires 1 part cement, 2 parts sand, and 1 part stones. Another kind requires 2 parts cement, 1 part sand, and 2 parts stones. The proportional costs are cement 3/- per part, sand 6d per part, and stones 2d per part. If we consider the expression $\begin{pmatrix}1 & 2 & 1\\ 2 & 1 & 2\end{pmatrix}\begin{pmatrix}36\\ 6\\ 2\end{pmatrix}$ and multiply these together in the same way as before, what does the answer represent?
(Do not worry that the numbers of elements in each row and column are different; use the general rule to do the multiplication as before. Your answer should be a vector.)

6. A housewife goes into a shop and buys 2 lb of sugar at 10d per lb, 2 lb of butter at 3/- per lb, ½ lb of coffee at 8/- per lb, and 1 lb of bacon at 4/6d per lb What do the arrays $(2\ \ 2\ \ \tfrac{1}{2}\ \ 1)$ and $\begin{pmatrix}\frac{5}{6}\\ 3\\ 8\\ 4\frac{1}{2}\end{pmatrix}$ have to do with her purchases? What have we found if we multiply

$$(2\ \ 2\ \ \tfrac{1}{2}\ \ 1).\begin{pmatrix}\frac{5}{6}\\ 3\\ 8\\ 4\frac{1}{2}\end{pmatrix}$$

7. Is it possible, using the multiplication rule as above, to find the product $\begin{pmatrix}1\\ 3\end{pmatrix}.\begin{pmatrix}4 & 5\\ 2 & 7\end{pmatrix}$?

8. Find the matrices represented by

(i) $\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}.\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$ and (ii) $\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}.\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$

Do you notice anything unusual about these results?

9. *a*, *b*, *c*, and *d* are any four numbers forming a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Find the matrices represented by

(i) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and (ii) $\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

If we compare matrix multiplication with ordinary multiplication of arithmetic, the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ behaves in the same way as which number?

10. Find the products $\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$ and $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}$. Again compare matrix multiplication with that of arithmetic. Which number behaves in the same way as $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$?

11. Find the product $\begin{pmatrix}4 & 2\\ 6 & 3\end{pmatrix}.\begin{pmatrix}1 & 5\\ -2 & -10\end{pmatrix}$. In what way is this result surprising?

***12.** $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a matrix such that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$. Find expressions for *c* and *d* in terms of *a* and *b*.

***13.** $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ is a matrix such that $\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Find expressions for *c* and *d* in terms of *a* and *b*.

***14.** Show that the product of two matrices of the form $\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}$ is also in the same form. How is this matrix multiplication connected with the ordinary multiplication of numbers?

***15.** Find the connections between *a*, *b*, *c*, and *d* if

$$\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}.\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}.\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix}$$

We now go back to consider why it is that the matrix which appeared at the end of column *I* on p. 34 enabled us to get the answer to the simultaneous equations in one matrix multiplication. We look at the matrices which appear in column *E*. Each one of these is obtained by taking the matrix $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and treating it in the same way as we treated the main matrix in the first column. If we now try using matrix multiplication, we see that these matrices in column E are matrices which transform those in the first column in the required way, for

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 9 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix} . \begin{pmatrix} 1 & -\frac{1}{2} \\ 9 & -4 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} . \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Altogether we have

$$\left[\begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ -9 & 1 \end{pmatrix} . \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\right] . \begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and working out the square bracket we get

$$\begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$$

The multiplication which we originally defined as a convenient rule for obtaining the equations from their matrix form has turned out to be closely involved in the process that leads to the solution of the equations. In the previous exercise we have seen that this same multiplication has everyday applications to bills and cost problems. In fact, it is an important mathematical process with many applications to physical problems. It is therefore important to study the behaviour (or the algebra) of these arrays of numbers.

We have seen already that it has several variations from the ordinary algebra of numbers. In what follows A, B, C, etc., will stand for matrices, all of which will be of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (called *two-by-two* matrices as they have two rows and two columns).

The first property that we have come across is that $A.B$ (. meaning matrix multiplication) is also a two-by-two matrix, but that $A.B$ may not be the same as $B.A$. We say that matrix multiplication is *not commutative*.

The second property is that there is a special matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which will always be denoted by I. This matrix is such that $I.A = A.I$ where A can be any other two-by-two matrix. This matrix behaves to matrix multiplication in the same way as 1 behaves to arithmetic. It is called the *unit matrix*.

The property that we want in order to get the solutions to simultaneous equations is that if we start with a matrix A, then we can obtain a matrix B such that $B.A = I$. In the case of our example A was $\begin{pmatrix} 2 & -1 \\ 9 & -4 \end{pmatrix}$ and B was $\begin{pmatrix} -4 & 1 \\ -9 & 2 \end{pmatrix}$. The method that was used to find B was rather long and tedious, and we will now show how a general formula can be obtained for getting at the value of B. We will look at the pattern of B as compared with A, and then investigate to see if this pattern was just coincidental. We will start with a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and, treating the a, b, c, and d in the same way as in the number case, we would get B as the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Now form the product $B.A$ which gives $\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$ and this will equal $A.B$ only if $ad-bc = 1$, and in general this will not be so.

(The original example happened to be one of the special cases which fitted this condition.) By choosing B to include a correcting factor, we can overcome this snag, and we find that if

$$B = \begin{pmatrix} \dfrac{d}{ad-bc} & \dfrac{-b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{pmatrix} \quad \text{then} \quad B.A = 1$$

This way of writing B is rather clumsy and we "take out the factor" $\dfrac{1}{ad-bc}$ and rewrite it as $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Notice that in order to "take out" a factor from a matrix, it must be a factor of all the number that make up the matrix.

The expression $(ad-bc)$ is called the *determinant* of the matrix A and is formed by taking the products of the terms diametrically opposite in the matrix, and then subtracting them. B is called the *inverse* of A. It is related to A in the same way that $\frac{1}{3}$ is related to 3, and you can think of it as the matrix which "unscrambles" the matrix of the original equations.

Now let us work an example of the solution of a pair of simultaneous equations using the matrix of the equations and its inverse.

Example

$$\begin{aligned} 4x+7y &= 1 \\ 5x-2y &= 3 \end{aligned}$$

which may be written in matrix form as

$$\begin{pmatrix} 4 & 7 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

The matrix A is therefore $\begin{pmatrix} 4 & 7 \\ 5 & -2 \end{pmatrix} \Rightarrow$ determinant* of $A = 4(-2) - 5 \times 7 = -43$. Therefore the inverse of A is

$$\frac{1}{-43} \cdot \begin{pmatrix} -2 & -7 \\ -5 & 4 \end{pmatrix}$$

and so the solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-43}\begin{pmatrix} -2 & -7 \\ -5 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = -\frac{1}{43}\begin{pmatrix} -23 \\ 7 \end{pmatrix}$$

i.e. $$x = \frac{23}{43} \quad \text{and} \quad y = -\frac{7}{43}$$

Notice that if the determinant of A is zero it is impossible to find the inverse of A. Such a matrix is called *singular*, and a pair of equations which has a singular matrix will not have solutions in the ordinary sense. Question 7 of Exercise 6 is an example illustrating this point. Any matrix which is not singular is called *non-singular*. All non-singular matrices have inverse matrices associated with them. The inverse of the matrix which so far we have denoted by B, is usually written as A^{-1}.

EXERCISE 6

1. Find the values of the determinant of A when $A =$

(i) $\begin{pmatrix} 7 & 1 \\ 2 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 5 & -1 \\ 2 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} -9 & -1 \\ 1 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} 3 & 6 \\ 2 & 4 \end{pmatrix}$ (v) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (vi) $\begin{pmatrix} -5 & -2 \\ 10 & 4 \end{pmatrix}$

---

* The symbol ⇔ means "implies and is implied by" while ⇒ means "implies".

The determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ may be written as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, but great care should be taken to distinguish between matrix and determinant. The determinant has straight lines at the sides.

2. Which of the matrices in question 1 are singular?

3. Find the inverse matrix of the following matrices:

(i) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} -5 & 7 \\ 1 & 2 \end{pmatrix}$

(iv) $\begin{pmatrix} -5 & -8 \\ 3 & -1 \end{pmatrix}$ (v) $\begin{pmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{4} \end{pmatrix}$

4. In the last question verify that $A.(A^{-1}) = (A^{-1}).A = 1$ in each case. (This result, which is always true, will be proved later.)

5. Solve the following pairs of simultaneous equations using the method of pages 42–3. Work out the transforming matrix for each stage of the manipulation, and verify that the product of all the transforming matrices is the inverse matrix.

(i) $3x-5y=-9$
$2x+7y=25$

(ii) $2x+3y=2$
$5x+15y=8$

6. Solve the following pairs of simultaneous equations by finding the inverse matrix in each case.

(i) $5x+2y=19$
$3x+y=11$

(ii) $7x+8y=5$
$2x+13y=6$

(iii) $3x-2y=11$
$5x-8y=4$

(iv) $x+3y=15$
$-2x+y=12$

7. Show that the matrix of the following pair of equations is singular. Draw their graphs and discover why it is that they do not have a solution.

$$\begin{aligned} 2x+3y &= 7 \\ 10x+15y &= 11 \end{aligned}$$

8. If $ax+by=c$ and $px+qy=r$ are two simultaneous equations in x and y, and a, b, c, p, q, and r are all whole numbers, then show that the solutions for x and y will also be whole numbers if $aq-bp=1$, whatever the values of c and d. This will also be true for another value of $aq-bp$. What is this value?

9. Use matrices and their inverses to find the solutions of the following simultaneous equations

(i) $\frac{3}{x}+\frac{2}{y}=7$
$\frac{5}{x}+\frac{1}{y}=11$

(ii) $x^2+y^2=25$
$3x^2-2y^2=30$

(iii) $\frac{7}{x-3}+\frac{2}{y-11}=2$
$\frac{2}{x-3}+\frac{7}{y-11}=6$

(iv) $5x^2+3xy=8$
$3x^2+2xy=-60$

From question 4 of the last set of examples, we have seen that the inverse matrix has the special property that it does not matter which "side" it is multiplied with its partner, the result is always I. In order to prove this we will assume that there are two possible different matrices X and Y such that $A.X=I$ and $Y.A=I$.

Then

$$A.X = I \Rightarrow Y.(A.X) = Y.I = Y \Rightarrow (Y.A).X = Y$$
$$\Rightarrow I.X = Y \Rightarrow X = Y$$

proving the result.

When we considered vectors in two dimensions earlier, we described them by using their components along two axes. It was convenient, but by no means necessary, to take these axes at right angles. In fact we could take any two different lines through O, the chosen origin, to act as axes, splitting the vectors up into components parallel to the chosen axes as in figure 15.

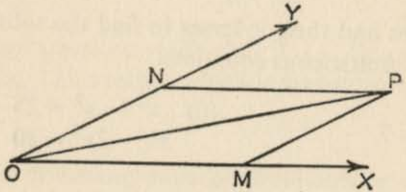

FIG. 15

We shall now see how we can change our description of a given vector using different axes. We shall start with a pair of axes at right angles and $\mathbf{i}$ and $\mathbf{j}$ as unit vectors along these axes. Now consider two other vectors, say $\mathbf{u} = 2\mathbf{i}+4\mathbf{j}$ and $\mathbf{v} = 3\mathbf{i}+5\mathbf{j}$. The problem is to take any other vector—say $\mathbf{w} = 7\mathbf{i}+2\mathbf{j}$ and express it in terms of $\mathbf{u}$ and $\mathbf{v}$. In order to do this we must find expressions for $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{u}$ and $\mathbf{v}$ and then substitute for $\mathbf{i}$ and $\mathbf{j}$ in the equation $\mathbf{w} = 7\mathbf{i}+2\mathbf{j}$, so finding the required expression in terms of $\mathbf{u}$ and $\mathbf{v}$. To get the expressions for $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{u}$ and $\mathbf{v}$ all we need to do is to solve the equations $\begin{matrix}\mathbf{u} = 2\mathbf{i}+4\mathbf{j}\\ \mathbf{v} = 3\mathbf{i}+5\mathbf{j}\end{matrix}$ simultaneously. We shall use matrices to do this,

writing the equations as $\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}2 & 4\\ 3 & 5\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}$ and then multiplying each side of the equations by $-\frac{1}{2}\begin{pmatrix}5 & -4\\ -3 & 2\end{pmatrix}$ which is the inverse of $\begin{pmatrix}2 & 4\\ 3 & 5\end{pmatrix}$. This then gives

$$-\tfrac{1}{2}\begin{pmatrix}5 & -4\\ -3 & 2\end{pmatrix}\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = -\tfrac{1}{2}\begin{pmatrix}5 & -4\\ -3 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\ 3 & 5\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix} = \begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}$$

By working out the left-hand side of this equation we finally get $\begin{pmatrix}-\frac{5}{2}\mathbf{u}+2\mathbf{v}\\ \frac{3}{2}\mathbf{u}-\mathbf{v}\end{pmatrix} = \begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}$ or in other words the equations $\mathbf{i} = -\frac{5}{2}\mathbf{u}+2\mathbf{v}$ and $\mathbf{j} = \frac{3}{2}\mathbf{u}-\mathbf{v}$. Consequently

$$\mathbf{w} = 7\mathbf{i}+2\mathbf{j} = 7(-\tfrac{5}{2}\mathbf{u}+2\mathbf{v})+2(\tfrac{3}{2}\mathbf{u}-\mathbf{v}) = -\tfrac{29}{2}\mathbf{u}+12\mathbf{v}$$

Similarly any other vector which is originally expressed in terms of $\mathbf{i}$ and $\mathbf{j}$ can also be expressed in terms of $\mathbf{u}$ and $\mathbf{v}$. Since all vectors in the plane of the axes could be expressed in terms of $\mathbf{i}$ and $\mathbf{j}$, it follows that they may also be expressed in terms of $\mathbf{u}$ and $\mathbf{v}$. Any set of vectors which have this property are said to *span* the plane. Notice that there must be at least two of them, for combinations of just one vector can give only vectors along its own direction. On the other hand, two are sufficient for the job, and if there are three or more the description will not be unique, for if a vector is given in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ then $\mathbf{w}$ can be expressed in terms of $\mathbf{u}$ and $\mathbf{v}$, and so we can get another description that uses only $\mathbf{u}$ and $\mathbf{v}$. In the case where there are just two distinct vectors used for describing all the other vectors in their plane, the two vectors are called a *basis* for the plane.

The work of the last paragraph is often called a *transformation* of vectors in the plane, for it can be thought of as starting with two vectors **i** and **j** along the lines $x = 0$ and $y = 0$, and transforming them into vectors $2\mathbf{i} + 4\mathbf{j}$ and $3\mathbf{i} + 5\mathbf{j}$ respectively. These new vectors lie along the lines $4x - 2y = 0$ and $3x - 5y = 0$ respectively, so that these two lines have now become the new axes of reference for **u** and **v**. Notice, too, that in the process the scales have been changed, for if **i** and **j** are each of length 1 unit, then $2\mathbf{i} + 4\mathbf{j}$ is of length $\sqrt{(2^2 + 4^2)} = \sqrt{20}$ whilst $3\mathbf{i} + 5\mathbf{j}$ is of length $\sqrt{(3^2 + 5^2)} = \sqrt{34}$. This transformation may be written in vector notation as $\begin{pmatrix} x \\ y \end{pmatrix}$ transforms into $\begin{pmatrix} a \\ b \end{pmatrix}$ where $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2x+4y \\ 3x+5y \end{pmatrix}$ or using matrices $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. This last equation is precisely the same as that which was used in the previous paragraph, and is the equation which enables us to work out what any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ will transform into using this particular transformation.

Some examples of the kinds of transformation that can be described using vectors appear in the following exercise. The varied applications of this property make vectors and matrices appear in many branches of mathematics.

EXERCISE 7

(Matrix methods should be used for example in this exercise).

1. If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = \mathbf{i} + \mathbf{j}$, and $\mathbf{w} = 5\mathbf{i} + 6\mathbf{j}$, express **w** in terms of **u** and **v**. If **i** and **j** are unit vectors along perpendicular axes OX and OY, use graph paper to illustrate what has happened as a result of doing this transformation.

2. If $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, and $\mathbf{w} = \mathbf{u} + \mathbf{v}$, express **i** and **j** in terms of (i) **u** and **w**, and (ii) **v** and **w**.

3. If $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $\mathbf{v} = 6\mathbf{i} - 3\mathbf{j}$ try to express **i** and **j** in terms of **u** and **v**. Draw a rough diagram to illustrate what is happening in this case. Do **u** and **v** form a basis for the vectors spanned by **i** and **j**?

4. You are given the transformation $\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} . \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}$. By considering a diagram or otherwise, find what kind of physical transformation of vectors this represents. How does this fit in with the definition of how to factorize the matrix that was given on p. 42.

5. By considering a diagram with various numerical values of a and b find the effect of the transformations $\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = A\begin{pmatrix} a \\ b \end{pmatrix}$ where A is the matrix:

(i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

(iii) $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

6. The four matrices A, B, C, and D are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ respectively. Work out all the possible products $A.B$, $B.D$, etc., and explain your answers in terms of the transformations that the matrices represent. (A set of matrices such as these is said to be *closed under multiplication*.) What are the inverses of each of these matrices?

7. We are given a transformation that can be expressed in matrix vector form and which changes $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ into $\begin{pmatrix} 13 \\ -3 \end{pmatrix}$

and also changes $\begin{pmatrix}2\\1\end{pmatrix}$ into $\begin{pmatrix}11\\-1\end{pmatrix}$. Find the matrix of the transformation that will transfer the second set of vectors back into the first set.

8. If we use a matrix transformation to transform $\begin{pmatrix}5\\2\end{pmatrix}$ into $\begin{pmatrix}21\\9\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ into $\begin{pmatrix}6\\3\end{pmatrix}$, what do we transform $\begin{pmatrix}2\\1\end{pmatrix}$ into? Which vector transforms into $\begin{pmatrix}27\\11\end{pmatrix}$?

9. We transform from $\begin{pmatrix}\mathrm{i}\\\mathrm{j}\end{pmatrix}$ into $\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}$ by the transformation $\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}=\begin{pmatrix}3&1\\7&7\end{pmatrix}\begin{pmatrix}\mathrm{i}\\\mathrm{j}\end{pmatrix}$. We then transform $\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}$ into $\begin{pmatrix}\mathrm{x}\\\mathrm{y}\end{pmatrix}$ by the transformation $\begin{pmatrix}\mathrm{x}\\\mathrm{y}\end{pmatrix}=\begin{pmatrix}2&1\\1&2\end{pmatrix}\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}$. What is the transformation (in matrix form) that transforms $\begin{pmatrix}\mathrm{i}\\\mathrm{j}\end{pmatrix}$ into $\begin{pmatrix}\mathrm{x}\\\mathrm{y}\end{pmatrix}$?

10. $\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}=\begin{pmatrix}3&2\\6&4\end{pmatrix}\begin{pmatrix}\mathrm{i}\\\mathrm{j}\end{pmatrix}$. Show that it is impossible to transform back from $\begin{pmatrix}\mathrm{u}\\\mathrm{v}\end{pmatrix}$ into $\begin{pmatrix}\mathrm{i}\\\mathrm{j}\end{pmatrix}$ both by considering matrices and by illustrating the transformation graphically.

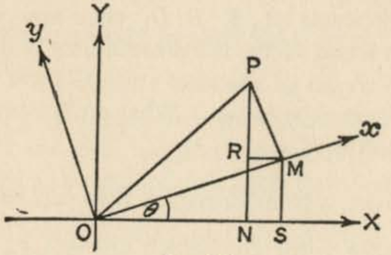

FIG. 16

11. Figure 16 represents a rotation of a pair of axes Ox, Oy (at right angles) through an angle $\theta°$ to new positions OX, OY respectively. The coordinates of P are (x, y) when referred to axes Ox, Oy and (X, Y) when referred to OX, OY. By considering PN = PR + RN = PR + MS, and a similar argument for the length of OS, show that the transformation which takes (x, y) into (X, Y) is

$$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

What is the transformation that represents a rotation through $\theta°$ in the reverse direction?

12. Using the data obtained in question 11 construct the matrix transformation that turns a figure through 120°. ABC is an equilateral triangle such that the origin is at the circumcentre of the triangle, and the base of the triangle is parallel to the x-axis. Find the coordinates of the vertices. (Take the circumradius as being of 1 unit.) Use the transformation to check that your answers are correct.

13. A right-angled triangle has its vertices at the points (2, 0), (0, 0), and (0, 1). Another right-angled triangle has its vertices at the points (−2, 0), (0, 0), and (0, 1). Show that it is impossible to find a matrix transformation of the kind investigated in question 11 which will transfer one triangle into the other, and find the physical explanation of why this is so.

14. What is the effect of the following transformation?

$$\begin{pmatrix}X\\Y\end{pmatrix}=\begin{pmatrix}0&-3\\3&0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

In particular if we have a triangle with vertices at the three points (a, b), (c, d), and (f, g), and these transfer into the points (A, B), (C, D), and (F, G), what geometrical relationship exists between the two triangles?

15. $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. There are two values of λ such that $\begin{pmatrix} x \\ y \end{pmatrix}$ will transform into $\begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$, i.e. $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \lambda x \\ \lambda y \end{pmatrix}$. Find the values of λ. Find also the directions of the vectors for which this is true. What physical effect does this vector property represent?

16. By considering the result of the transformation $\begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix}$ operating on the vector $\begin{pmatrix} \cos A \\ \sin A \end{pmatrix}$ show that:

(i) $\cos(A+B) = \cos A \cos B - \sin A \sin B$

(ii) $\sin(A+B) = \sin A \cos B + \cos A \sin B$.

17. Find the matrices that:

(i) turn a vector through 45°.

(ii) reflect a vector in the x-axis.

(iii) reflect a vector in the line $y = x$.

(Reflection is defined in question 9 of Exercise 4.)

18. P is the point $(3, 0)$, Q is the point $(0, 4)$. OP and OQ are transformed by the transformation represented by $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$ into vectors OP' and OQ' respectively. Find the ratio of the areas of the triangle OPQ to the triangle OP′Q′. How is this ratio connected with the determinant of the transformation? (Try to prove your answer to be true whatever two vectors OP and OQ you start with.)

Answers

EXERCISE 1

3. (i) (12) (ii) $\frac{1}{2}(a+b)$

4. (i) $(\frac{24}{7})$ (ii) $(-\frac{4}{7})$ (iii) $(-\frac{27}{7})$ (iv) $\frac{1}{7}(4a+3b)$

5. 5i

6. No.

7. All points between (2) and (5).

9. $(\frac{1}{5})$

10. $(-\frac{1}{3})$

11. (i) $\left(\dfrac{p}{p+1}\right)$ (ii) $\left(\dfrac{p}{p-1}\right)$ (iii) Yes

(iv) In Q.9 X could not be between O and P and in Q.10 X could not be on PO produced.

(v) In Q.9 X is the mid-point of OP and in Q.10 it has moved infinitely far along PO produced.

12. $\dfrac{r-q}{r(1-q)}$

13. 1 or -3. 1 makes P and R the same point.

14. (i) Y moves from I to a point infinitely far along OI produced.

(ii) Y start to move in from a point infinitely far along IO produced to O.

(iii) Y will be moving faster in each case the further it is away from O.

15. (i) Y moves from I to (4).

(ii) Y moves from (4) in the opposite direction to X.

(iii) Y moves at a speed equal to that of X, but in the opposite direction.

16. (i) As X moves from I to (-1), Y moves from I to a point infinitely far along OI produced with increasing speed. If X then continues, Y starts moving back from a point infinitely far along IO produced.

(ii) Y moves from I to (-1) at a decreasing speed.

(iii) (-4), (2).

17. $3xy-2x-4y+4 = 0$

EXERCISE 2

1. 4·67 units at an angle of 28° to OP.

2. 11·40 units.

3. 2·19 units at an angle of 91° 30′ to OP.

5. 4·11 units at an angle of 76° to OP.

6. 8·07 units at an angle of 29° 40′ to OP.

8. 8·61 lbf at 48° 30′ to the direction of the 3 lbf.

9. 403 mile/h at N 7° 7′ E.

10. (i) 53° 8′ to the bank, going upstream (ii) $\frac{1}{2}$ hour.

EXERCISE 3

1. 16·64

2. 13·93; 68° 58′

3. 10·05

4. 2·828; 45°

5. 8·25; 75° 58′ to the x-axis.

6. $\begin{pmatrix} 6{\cdot}55 \\ 4{\cdot}59 \end{pmatrix}$

7. $-\mathbf{i} - 4\mathbf{j}$·

8. 14·42; 35° 27′ to the x-axis.

9. 8·06; 29° 45′ to the x-axis.

10. 13; 67° 23′ to the x-axis.

11. $x = 2; y = -5$

12. $x = \frac{1}{2}; y = -4$

13. 5; 36° 52′ to the x-axis.

15. (i) 13° 27′ (ii) 90° 46′ (iii) 89° 13′ (iv) 13° 27′

EXERCISE 4

1. Yes **2.** Yes **3.** (i) No (ii) No

4. (i) 9 (ii) $\frac{16}{3}$ (iii) $\frac{73}{5}$ **5.** 5 **6.** 5·83

7. 16·12

8. (i) $(2, \frac{9}{2})$ (ii) $\left(\frac{a+c}{2}, \frac{b+d}{2}\right)$

9. (i) $(3, -4)$ (ii) $(-3, 4)$ (iii) $(4, 3)$

10. $(-2, 3)$ **11.** $\frac{1}{2}$ **12.** $\frac{2}{3}$ **13.** $(3, 2)$ **14.** 15

EXERCISE 5

1. (i) $\begin{aligned} 3x-y &= 7 \\ -2x+4y &= 2 \end{aligned}$ (ii) $\begin{aligned} 7x+2y &= 11 \\ 5x-2y &= 5 \end{aligned}$

2. (i) $\begin{pmatrix} 27 \\ 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ (iii) $\begin{pmatrix} -2 \\ -11 \end{pmatrix}$ (iv) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

3. (i) $\begin{pmatrix} 11 & 17 \\ 23 & 37 \end{pmatrix}$ (ii) $\begin{pmatrix} -4 & -1 \\ 1 & 0 \end{pmatrix}$ (iii) $\begin{pmatrix} 16 & 40 \\ 16 & 40 \end{pmatrix}$
(iv) $\begin{pmatrix} 58 & 42 \\ 42 & 58 \end{pmatrix}$

4. Total amounts they each spend.

5. Total costs in pence of the different kind of concrete per part.

6. (ii) Total amount of money that she spends.

7. No.

8. (i) $\begin{pmatrix} 3 & 6 \\ 7 & 14 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 & 10 \\ 7 & 10 \end{pmatrix}$ The answers differ.

9. (i) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (ii) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (iii) 1

10. (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (iii) 0

11. (i) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (ii) the product of two non-zero numbers cannot be zero.

12. $c = -a^2/b;\ d = -a$

13. $c = \dfrac{1-a^2}{b};\ d = -a$

15. $a = d;\ b = c$

EXERCISE 6

1. (i) 19 (ii) 17 (iii) -17 (iv) 0 (v) 1 (vi) 0

2. (iv) (vi)

3. (i) $-\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (iii) $-\frac{1}{17}\begin{pmatrix} 2 & -7 \\ -1 & -5 \end{pmatrix}$
(iv) $\frac{1}{29}\begin{pmatrix} -1 & 8 \\ -3 & -5 \end{pmatrix}$ (v) $-120\begin{pmatrix} \frac{1}{4} & -\frac{2}{3} \\ -\frac{1}{5} & \frac{1}{2} \end{pmatrix}$

5. (i) $x = 2, y = 3$ (ii) $x = \frac{2}{5};\ y = \frac{2}{5}$

6. (i) $x = 3, y = 2$ (ii) $x = \frac{17}{75}, y = \frac{32}{75}$
(iii) $x = \frac{40}{7}, y = \frac{43}{14}$ (iv) $x = -3, y = 6$

8. -1

9. (i) $x = \frac{7}{15}, y = \frac{7}{2}$ (ii) $x = \pm 4;\ y = \pm 3$
(iii) $x = \frac{51}{2}, y = \frac{463}{38}$ (iv) $x = \pm 14;\ y = \mp 23\frac{1}{7}$

EXERCISE 7

1. $\mathbf{w} = \mathbf{u} + 3\mathbf{v}$

2. (i) $\mathbf{i} = \frac{1}{29}(3\mathbf{u} + 4\mathbf{w});\ \mathbf{j} = \frac{1}{29}(5\mathbf{u} - 3\mathbf{w})$
(ii) $\mathbf{i} = 3\mathbf{v} - \mathbf{w};\ \mathbf{j} = 5\mathbf{v} - 2\mathbf{w}$

3. No.

6. Inverses are $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ respectively.

7. $\frac{1}{20}\begin{pmatrix} 5 & 15 \\ 1 & -9 \end{pmatrix}$

8. (i) $\begin{pmatrix} 9 \\ 4 \end{pmatrix}$ (ii) $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$

9. $\begin{pmatrix} 13 & 9 \\ 17 & 15 \end{pmatrix}$

11. $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

12. $\begin{pmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix}$

A is $(0, 1)$ B is $\left(\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$ C is $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$

ANSWERS

13. Transformation requires reflection.

14. Rotation through 90° in anti-clockwise sense, and magnification of 3 times. Similar triangles in the ratio 1 : 3.

15. $\lambda = -2$ or 5; $\begin{pmatrix}-4\\3\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}$

17. (i) $\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$ (ii) $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ (iii) $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$

18. 1 : 11.